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Equivalent Measures of Dependence*

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The maximal correlation between a pair of σ-fields $\mathscr{A}$ and $\mathscr{B}$ becomes arbitrarily small as $\sup\{|P(A \cap B) - P(A)P(B)|/[P(A)P(B)]^{1/2},\ A \in \mathscr{A},\ B \in \mathscr{B},\ P(A) > 0,\ P(B) > 0\}$ becomes sufficiently small.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and for any event $F \in \mathscr{F}$ let I_F denote its indicator function. For any two r.v.'s X and Y with $EX^2 < \infty$ and $EY^2 < \infty$ one defines the correlation

$$\operatorname{corr}(X, Y) = \frac{E[(X - EX)(Y - EY)]}{E^{1/2}(X - EX)^2\, E^{1/2}(Y - EY)^2}$$

with $\operatorname{corr}(X, Y) \equiv 0$ if X or Y is constant a.s. Define the following measures of dependence between σ-fields $\mathscr{A}$ and $\mathscr{B} \subset \mathscr{F}$:

$$\begin{aligned}
\lambda(\mathscr{A}, \mathscr{B}) &= \sup |P(A \cap B) - P(A)\,P(B)|/[P(A)\,P(B)]^{1/2}, \\
&\qquad A \in \mathscr{A}, \quad B \in \mathscr{B}, \quad P(A) > 0, \quad P(B) > 0, \\
\tau(\mathscr{A}, \mathscr{B}) &= \sup |\operatorname{Corr}(I_A, I_B)|, \\
&\qquad A \in \mathscr{A}, \quad B \in \mathscr{B}, \\
\rho(\mathscr{A}, \mathscr{B}) &= \sup |\operatorname{Corr}(f, g)|, \\
&\qquad f \in \mathscr{L}^2(\mathscr{A}), \quad g \in \mathscr{L}^2(\mathscr{B}).
\end{aligned}$$

The quantity $\rho(\mathscr{A}, \mathscr{B})$ is the well-known "maximal correlation" between $\mathscr{A}$ and $\mathscr{B}$ [2]. The purpose of this paper is to show that these three measures of

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dependence are "equivalent" in the following sense: by having any one of them sufficiently small one can make the other two arbitrarily small.

It is easy to show that the inequalities $0 \leqslant \lambda(\mathscr{A}, \mathscr{B}) \leqslant \tau(\mathscr{A}, \mathscr{B}) \leqslant \rho(\mathscr{A}, \mathscr{B}) \leqslant 1$ and $\tau(\mathscr{A}, \mathscr{B}) \leqslant 2\lambda(\mathscr{A}, \mathscr{B})$ always hold. Our task is completed once we prove

THEOREM 1. *For any probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and any two σ-fields $\mathscr{A}$ and $\mathscr{B} \subset \mathscr{F}$, $\rho(\mathscr{A}, \mathscr{B}) \leqslant 13[\tau(\mathscr{A}, \mathscr{B})]^{1/31}$.*

This inequality is not close to being sharp; instead "sharpness" is sacrificed in order to keep the proof simple.

Theorem 1 has the following simple application in stochastic processes: Suppose $(X_k,\ k = ..., -1, 0, 1,...)$ is a sequence of random variables, and for each integer J let $\mathscr{P}_J$ (resp. $\mathscr{F}_J$) denote the σ-field generated by the block of r.v.'s $(X_k,\ k \leqslant J)$ (resp. $(X_k,\ k \geqslant J)$). Then the following three mixing conditions are equivalent:

$$\lambda(n) \equiv \sup_J \lambda(\mathscr{P}_J, \mathscr{F}_{J+n}) \to 0 \quad \text{as} \quad n \to \infty,$$

$$\tau(n) \equiv \sup_J \tau(\mathscr{P}_J, \mathscr{F}_{J+n}) \to 0 \quad \text{as} \quad n \to \infty,$$

$$\rho(n) \equiv \sup_J \rho(\mathscr{P}_J, \mathscr{F}_{J+n}) \to 0 \quad \text{as} \quad n \to \infty.$$

The latter one $\rho(n) \to 0$ has been studied in many articles; see, for example, [1, 3, 4].

Proof of Theorem 1. Assume $\mathscr{A}$ and $\mathscr{B}$ are σ-fields, and let $t = \tau(\mathscr{A}, \mathscr{B})$. If $t = 0$ or $t \geqslant 13^{-31}$, then Theorem 1 is trivial; so we assume

$$0 < t < 13^{-31}. \tag{1}$$

Suppose f and g are r.v.'s of the form

$$f = \sum_{i=1}^{I} f_i I_{A(i)}, \qquad g = \sum_{j=1}^{J} g_j I_{B(j)},$$

where the following conditions are satisfied: $I \geqslant 2$ and $J \geqslant 2$; $\{A(1),..., A(I)\}$ and $\{B(1),..., B(J)\}$ are each a partition of Ω; $\forall i$, $A(i) \in \mathscr{A}$ and $P(A(i)) > 0$; $\forall j$, $B(j) \in \mathscr{B}$ and $P(B(j)) > 0$; $f_1 < f_2 < \cdots < f_I$ and $g_1 < g_2 < \cdots < g_J$ are real numbers; $Ef = Eg = 0$ and (as is implied by the above conditions) $\operatorname{var} f > 0$ and $\operatorname{var} g > 0$. Modulo addition by constants, such r.v.'s are dense in $\mathscr{L}^2(\mathscr{A})$ and $\mathscr{L}^2(\mathscr{B})$, respectively, and hence to prove Theorem 1 it suffices to prove $|\operatorname{corr}(f, g)| \leqslant 13t^{1/31}$. Now the r.v. $-f$ can be expressed in the same way as f, and $\operatorname{corr}(-f, g) = -\operatorname{corr}(f, g)$, so to prove Theorem 1 it suffices to prove

$$\operatorname{corr}(f, g) \leqslant 13t^{1/31}. \tag{2}$$

For each $i = 1,..., I-1$ define the event $C(i) = \bigcup_{k=1}^{i} A(k)$, the positive numbers $q_i \equiv f_{i+1} - f_i$ and $c_i \equiv P(C(i))$, and the r.v. $V_i \equiv c_i - I_{C(i)}$. For each $j = 1,..., J-1$ define the event $D(j) \equiv \bigcup_{l=1}^{j} B(l)$, the positive numbers $r_j \equiv g_{j+1} - g_j$ and $d_j \equiv P(D(j))$, and the r.v. $W_j \equiv d_j - I_{D(j)}$. Note that $EV_i = 0$, $\forall i$ and $EW_j = 0$, $\forall j$. Keeping in mind that $Ef = Eg = 0$ by assumption, one can easily show that

$$\sum_{i=1}^{I-1} q_i V_i = f, \qquad \sum_{j=1}^{J-1} r_j W_j = g. \tag{3}$$

We need to define two functions on the unit square $[0, 1] \times [0, 1]$:

$$M(x, y) = \min\{x(1-y), y(1-x)\},$$
$$H(x, y) = \min\{M(x, y), t[x(1-x)\, y(1-y)]^{1/2}\}.$$

For each i and j, $\operatorname{corr}(V_i, W_j) = \operatorname{corr}(I_{C(i)}, I_{D(j)}) \leqslant t$, and by taking into account the additional restriction that $P(C(i) \cap D(j)) \leqslant \min\{P(C(i)), P(D(j))\}$ we find that $EV_i W_j \leqslant H(c_i, d_j)$. Hence by (3),

$$Efg \leqslant \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} q_i r_j H(c_i, d_j).$$

Also it is easy to show that if $1 \leqslant i, k \leqslant I-1$, then $EV_i V_k = M(c_i, c_k)$, and if $1 \leqslant j, l \leqslant J-1$, then $EW_j W_l = M(d_j, d_l)$. Hence by (3),

$$Ef^2 = \sum_{i=1}^{I-1} \sum_{k=1}^{I-1} q_i q_k M(c_i, c_k), \qquad Eg^2 = \sum_{j=1}^{J-1} \sum_{l=1}^{J-1} r_j r_l M(d_j, d_l).$$

For each $\delta > 0$ define the functions $a_\delta(\cdot)$ and $b_\delta(\cdot)$ on $[0, 1]$ by

$$a_\delta(x) = (2\delta)^{-1} \sum_{i=1}^{I-1} q_i I_{[c(i)-\delta, c(i)+\delta]}(x),$$

$$b_\delta(x) = (2\delta)^{-1} \sum_{j=1}^{J-1} r_j I_{[d(j)-\delta, d(j)+\delta]}(x),$$

where $c(i) \equiv c_i$ and $d(j) \equiv d_j$. Note that these functions are nonnegative.

From the definitions above, we have $0 < c_i < 1$, $\forall i$, $1 \leqslant i \leqslant I-1$, and

$0 < d_j < 1$, $\forall j$, $1 \leqslant j \leqslant J-1$. Also $H(x, y)$ and $M(x, y)$ are each continuous on $[0, 1] \times [0, 1]$. It follows that

$$\lim_{\delta \downarrow 0} \int_0^1 \int_0^1 a_\delta(x)\, b_\delta(y)\, H(x, y)\, dy\, dx = \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} q_i r_j H(c_i, d_j) \geqslant Efg,$$

$$\lim_{\delta \downarrow 0} \int_0^1 \int_0^1 a_\delta(x)\, a_\delta(y)\, M(x, y)\, dy\, dx = Ef^2,$$

$$\lim_{\delta \downarrow 0} \int_0^1 \int_0^1 b_\delta(x)\, b_\delta(y)\, M(x, y)\, dy\, dx = Eg^2.$$

Hence to prove Theorem 1 it suffices to prove the following technical lemma (see (2)):

LEMMA 0. *Suppose $a(\cdot)$ and $b(\cdot)$ are nonnegative integrable Borel functions on* $[0, 1]$, *then*

$$\int_0^1 \int_0^1 a(x)\, b(y)\, H(x, y)\, dy\, dx \leqslant 13t^{1/31} \left[\int_0^1 \int_0^1 a(x)\, a(y)\, M(x, y)\, dy\, dx\right]^{1/2} \times \left[\int_0^1 \int_0^1 b(x)\, b(y)\, M(x, y)\, dy\, dx\right]^{1/2}.$$

The rest of this article is devoted to proving Lemma 0. First define the parameter $u = t^{6/31}$.

LEMMA 1. *Suppose $0 \leqslant x, y \leqslant 1$ and all of the following conditions hold*: (i) $x \geqslant uy$, (ii) $y \geqslant ux$, *(iii)* $(1-x) \geqslant u(1-y)$, *and* (iv) $(1-y) \geqslant u(1-x)$. *Then $H(x, y) \leqslant (t/u)\, M(x, y)$.*

Proof. Without loss of generality we assume $0 \leqslant x \leqslant y \leqslant 1$. From (i) and (iv) we then have $u^2 y(1-x) \leqslant x(1-y) = M(x, y)$. Hence $H(x, y) \leqslant t[x(1-x)\, y(1-y)]^{1/2} \leqslant (t/u)\, M(x, y)$, and Lemma 1 is proved.

Let R denote the subset of $[0, 1] \times [0, 1]$ determined by the four inequalities (i)–(iv) in Lemma 1. Subset R consists of a rhombus and its interior. To prove Lemma 0, we shall first get an upper bound for $\iint a(x)\, b(y)\, M(x, y)\, dy\, dx$ when the integral is taken over R (Lemma 2), and another upper bound when the integral is taken over the complement $([0, 1] \times [0, 1]) - R$ (Lemma 3). At the very end we shall combine Lemmas 1–3 to verify Lemma 0 (and thus complete the proof of Theorem 1).

For any two nonnegative integrable Borel functions $a(\cdot)$ and $b(\cdot)$ on $[0, 1]$ and any Borel subset S of $[0, 1] \times [0, 1]$ define the integral

$$G_{ab}(S) = \iint_{(x,y) \in S} a(x)\, b(y)\, M(x, y)\, dy\, dx.$$

The unit square $[0, 1] \times [0, 1]$ itself will henceforth be denoted SQU.

LEMMA 2. *If $a(\cdot)$ and $b(\cdot)$ are nonnegative integrable Borel functions on $[0, 1]$, then*

$$G_{ab}(R) \leqslant 2u^{-4}[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}.$$

Proof. By (1) and the definition of u, $u < 13^{-6}$. Define the following subintervals $\gamma(n)$, $n = ..., -1, 0, 1,...$ of $[0, 1]$:

$$\begin{aligned} \gamma(n) &= [u^{n+2}, u^n], && \text{if } n \geqslant 1, \\ &= [u^2, 1 - u^2], && \text{if } n = 0, \\ &= [1 - u^{-n}, 1 - u^{-n+2}], && \text{if } n \leqslant -1. \end{aligned}$$

It is easy to show that $R \subset \{(0, 0), (1, 1)\} \cup (\bigcup_{n=-\infty}^{\infty} [\gamma(n) \times \gamma(n)])$. Also it is easy to show that for each fixed n, if $(x, y) \in \gamma(n) \times \gamma(n)$, then $M(x, y) \geqslant u^4 \max\{x(1 - y), y(1 - x)\}$. (If $n \neq 0$, then the u^4 can even be replaced by $u^2(1 - u^{|n|})$.)

For each fixed n,

$$\begin{aligned} G_{ab}(\gamma(n) \times \gamma(n)) &= [[G_{ab}(\gamma(n) \times \gamma(n))]^{1/2}]^2 \\ &\leqslant \left[\int_{\gamma(n)} \int_{\gamma(n)} a(x)\, b(y)\, x(1 - y)\, dy\, dx\right]^{1/2} \\ &\quad \times \left[\int_{\gamma(n)} \int_{\gamma(n)} a(w)\, b(z)\, z(1 - w)\, dz\, dw\right]^{1/2} \\ &= \left[\int_{\gamma(n)} \int_{\gamma(n)} a(w)\, a(x)(1 - w)\, x\, dx\, dw\right]^{1/2} \\ &\quad \times \left[\int_{\gamma(n)} \int_{\gamma(n)} b(y)\, b(z)(1 - y)\, z\, dz\, dy\right]^{1/2} \\ &\leqslant u^{-4}[G_{aa}(\gamma(n) \times \gamma(n))\, G_{bb}(\gamma(n) \times \gamma(n))]^{1/2}. \end{aligned}$$

When $|m-n| \geqslant 2$ the squares $\gamma(m) \times \gamma(m)$ and $\gamma(n) \times \gamma(n)$ do not overlap (except at corner points when $|m-n|=2$), and hence

$$\sum_{n=-\infty}^{\infty} G_{aa}(\gamma(n) \times \gamma(n)) = \left(\sum_{n \text{ even}} + \sum_{n \text{ odd}}\right) G_{aa}(\gamma(n) \times \gamma(n)) \leqslant 2G_{aa}(\text{SQU}).$$

Similarly $\sum_{n=-\infty}^{\infty} G_{bb}(\gamma(n) \times \gamma(n)) \leqslant 2G_{bb}(\text{SQU})$. Hence by Cauchy's inequality,

$$\begin{aligned} G_{ab}(R) &\leqslant \sum_{n=-\infty}^{\infty} G_{ab}(\gamma(n) \times \gamma(n)) \\ &\leqslant u^{-4} \sum_{n=-\infty}^{\infty} [G_{aa}(\gamma(n) \times \gamma(n))\, G_{bb}(\gamma(n) \times \gamma(n))]^{1/2} \\ &\leqslant u^{-4} \left[\sum_{n=-\infty}^{\infty} G_{aa}(\gamma(n) \times \gamma(n))\right]^{1/2} \left[\sum_{n=-\infty}^{\infty} G_{bb}(\gamma(n) \times \gamma(n))\right]^{1/2} \\ &\leqslant 2u^{-4}[G_{aa}(\text{SQU})\, G_{bb}(\text{SQU})]^{1/2} \end{aligned}$$

and Lemma 2 is proved.

Define the set $Q = (\text{SQU}) - R$.

LEMMA 3. *If $a(\cdot)$ and $b(\cdot)$ are nonnegative integrable Borel functions on* $[0, 1]$, *then*

$$G_{ab}(Q) \leqslant 11u^{1/6}[G_{aa}(\text{SQU})\, G_{bb}(\text{SQU})]^{1/2}.$$

Proof. Let $\varepsilon = u^{1/6} = t^{1/31}$. By (1) we have $\varepsilon < \frac{1}{13}$, an inequality that will be used later (without being mentioned again).

Let us define the following closed regions in SQU:

$$\beta(1) = [1-\varepsilon, 1] \times [0, \varepsilon], \qquad \beta(2) = [0, \varepsilon] \times [1-\varepsilon, 1],$$

$$\beta(3) = [\varepsilon, 1-\varepsilon] \times [0, \varepsilon^3], \qquad \beta(4) = [0, \varepsilon^3] \times [\varepsilon, 1'-\varepsilon],$$

$$\beta(5) = [\varepsilon, 1-\varepsilon] \times [1-\varepsilon^3, 1], \qquad \beta(6) = [1-\varepsilon^3, 1] \times [\varepsilon, 1-\varepsilon],$$

$$\beta(7) = \{(x, y)\colon 0 \leqslant x \leqslant \varepsilon,\ 0 \leqslant y \leqslant ux\},$$

$$\beta(8) = \{(x, y)\colon 0 \leqslant y \leqslant \varepsilon,\ 0 \leqslant x \leqslant uy\},$$

$$\beta(9) = \{(x, y)\colon 0 \leqslant 1-x \leqslant \varepsilon,\ 0 \leqslant 1-y \leqslant u(1-x)\},$$

$$\beta(10) = \{(x, y)\colon 0 \leqslant 1-y \leqslant \varepsilon,\ 0 \leqslant 1-x \leqslant u(1-y)\}.$$

It is easy to show that $Q \subset \bigcup_{k=1}^{10} \beta(k)$ and hence $G_{ab}(Q) \leqslant \sum_{k=1}^{10} G_{ab}(\beta(k))$. To prove Lemma 3 we shall get an upper bound on each of the numbers $G_{ab}(\beta(k))$.

To start with $\beta(1)$,

$$\begin{aligned}
G_{ab}(\beta(1)) &= \int_{1-\varepsilon}^{1} \int_{0}^{\varepsilon} a(x)\, b(y)(1-x)\, y\, dy\, dx \\
&= \left[\int_{1-\varepsilon}^{1} \int_{1-\varepsilon}^{1} a(x)\, a(w)(1-x)(1-w)\, dw\, dx\right]^{1/2} \\
&\quad \times \left[\int_{0}^{\varepsilon} \int_{0}^{\varepsilon} b(y)\, b(z)\, yz\, dz\, dy\right]^{1/2} \\
&\leqslant \left[\int_{1-\varepsilon}^{1} \int_{1-\varepsilon}^{1} a(x)\, a(w)[\varepsilon/(1-\varepsilon)]\, M(x, w)\, dw\, dx\right]^{1/2} \\
&\quad \times \left[\int_{0}^{\varepsilon} \int_{0}^{\varepsilon} b(y)\, b(z)[\varepsilon/(1-\varepsilon)]\, M(y, z)\, dz\, dy\right]^{1/2} \\
&\leqslant [\varepsilon/(1-\varepsilon)] \cdot [G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}.
\end{aligned}$$

The inequality $G_{ab}(\beta(2)) \leqslant [\varepsilon/(1-\varepsilon)][G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}$ can be established by a similar argument, and since $2\varepsilon/(1-\varepsilon) < 3\varepsilon$ we have

$$G_{ab}(\beta(1)) + G_{ab}(\beta(2)) \leqslant 3\varepsilon[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}. \tag{4}$$

Next we shall get a bound on $G_{ab}(\beta(3))$.

$$\begin{aligned}
G_{ab}(\beta(3)) &= \left[\int_{\varepsilon}^{1-\varepsilon} \int_{0}^{\varepsilon^3} a(x)\, b(y)(1-x)\, y\, dy\, dx\right] \\
&= \left[\int_{\varepsilon}^{1-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} a(w)\, a(x)(1-w)(1-x)\, dw\, dx\right]^{1/2} \\
&\quad \times \left[\int_{0}^{\varepsilon^3} \int_{0}^{\varepsilon^3} b(y)\, b(z)\, yz\, dz\, dy\right]^{1/2} \\
&\leqslant \left[\int_{\varepsilon}^{1-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} a(w)\, a(x)[(1-\varepsilon)/\varepsilon]\, M(w, x)\, dw\, dx\right]^{1/2} \\
&\quad \times \left[\int_{0}^{\varepsilon^3} \int_{0}^{\varepsilon^3} b(y)\, b(z)[\varepsilon^3/(1-\varepsilon^3)]\, M(y, z)\, dz\, dy\right]^{1/2} \\
&\leqslant \varepsilon[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2},
\end{aligned}$$

since $(1-\varepsilon)/(1-\varepsilon^3) \leqslant 1$. The inequality $G_{ab}(\beta(k)) \leqslant \varepsilon[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}$ can be established for $k = 4, 5, 6$ by similar arguments. Hence

$$\sum_{k=3}^{6} G_{ab}(\beta(k)) \leqslant 4\varepsilon[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}. \tag{5}$$

Now we shall get an upper bound on $G_{ab}(\beta(7))$; the argument for this will be a little longer than for $G_{ab}(\beta(k))$, $1 \leqslant k \leqslant 6$. Some preliminary work has to be done first.

For each $n = 1, 2, 3, \ldots$, let

$$a_n = \int_{\varepsilon^{n+1}}^{\varepsilon^n} a(x)\, dx, \qquad b_n = \int_{\varepsilon^{n+1}}^{\varepsilon^n} b(x)\, dx.$$

If $n \geqslant 1$ and $(x, y) \in [\varepsilon^{n+1}, \varepsilon^n] \times [\varepsilon^{n+1}, \varepsilon^n]$, then $M(x, y) \geqslant \varepsilon^{n+1}(1-\varepsilon)$. Hence for each $n \geqslant 1$,

$$\begin{aligned} &G_{aa}([\varepsilon^{n+1}, \varepsilon^n] \times [\varepsilon^{n+1}, \varepsilon^n]) \\ &\qquad \geqslant \varepsilon^{n+1}(1-\varepsilon) \int_{\varepsilon^{n+1}}^{\varepsilon^n} \int_{\varepsilon^{n+1}}^{\varepsilon^n} a(x)\, a(y)\, dy\, dx = \varepsilon^{n+1}(1-\varepsilon)\, a_n^2 \end{aligned}$$

and since the squares $[\varepsilon^{n+1}, \varepsilon^n] \times [\varepsilon^{n+1}, \varepsilon^n]$, $n = 1, 2, \ldots$, do not overlap (except at corner points) we have $\sum_{n=1}^{\infty} G_{aa}([\varepsilon^{n+1}, \varepsilon^n] \times [\varepsilon^{n+1}, \varepsilon^n]) \leqslant G_{aa}(\mathrm{SQU})$ and hence

$$\sum_{n=1}^{\infty} \varepsilon^n a_n^2 \leqslant \varepsilon^{-1}(1-\varepsilon)^{-1}\, G_{aa}(\mathrm{SQU}). \tag{6}$$

An analogous argument also gives

$$\sum_{n=1}^{\infty} \varepsilon^n b_n^2 \leqslant \varepsilon^{-1}(1-\varepsilon)^{-1}\, G_{bb}(\mathrm{SQU}). \tag{7}$$

Now if $1 \leqslant m < n$, then

$$\begin{aligned} &G_{ab}([\varepsilon^{m+1}, \varepsilon^m] \times [\varepsilon^{n+1}, \varepsilon^n]) \\ &\qquad = \int_{\varepsilon^{m+1}}^{\varepsilon^m} \int_{\varepsilon^{n+1}}^{\varepsilon^n} a(x)\, b(y)\, y(1-x)\, dy\, dx \leqslant \varepsilon^n a_m b_n \end{aligned}$$

It is easy to show that

$$\beta(7) \subset ([0, \varepsilon] \times \{0\}) \cup \left(\bigcup_{m=1}^{\infty} \bigcup_{n=m+6}^{\infty} ([\varepsilon^{m+1}, \varepsilon^m] \times [\varepsilon^{n+1}, \varepsilon^n]) \right)$$

and since $G_{ab}([0, \varepsilon] \times \{0\}) = 0$ we have

$$\begin{aligned}
G_{ab}(\beta(7)) &\leqslant \sum_{m=1}^{\infty} \sum_{n=m+6}^{\infty} G_{ab}([\varepsilon^{m+1}, \varepsilon^m] \times [\varepsilon^{n+1}, \varepsilon^n]) \\
&\leqslant \sum_{m=1}^{\infty} \sum_{n=m+6}^{\infty} \varepsilon^n a_m b_n \\
&= \sum_{m=1}^{\infty} \sum_{n=m+6}^{\infty} \varepsilon^{(n-m)/2} [\varepsilon^m a_m^2 \varepsilon^n b_n^2]^{1/2} \\
&= \sum_{m=1}^{\infty} \sum_{l=6}^{\infty} \varepsilon^{l/2} [\varepsilon^m a_m^2 \varepsilon^{m+l} b_{m+l}^2]^{1/2} \\
&\leqslant \sum_{l=6}^{\infty} \left(\varepsilon^{l/2} \left[\sum_{m=1}^{\infty} \varepsilon^m a_m^2 \right]^{1/2} \left[\sum_{m=1}^{\infty} \varepsilon^{m+l} b_{m+l}^2 \right]^{1/2} \right) \\
&\leqslant \sum_{l=6}^{\infty} (\varepsilon^{l/2} \varepsilon^{-1} (1-\varepsilon)^{-1} [G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2})
\end{aligned}$$

by (6) and (7). Since $\sum_{l=6}^{\infty} \varepsilon^{l/2} \varepsilon^{-1} (1-\varepsilon)^{-1} = [\varepsilon^3/(1-\varepsilon^{1/2})]$ $\varepsilon^{-1}(1-\varepsilon)^{-1} < \varepsilon$ we have

$$G_{ab}(\beta(7)) \leqslant \varepsilon [G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}.$$

By similar arguments one can establish $G_{ab}(\beta(k)) \leqslant \varepsilon [G_{aa}(\mathrm{SQU})$ $G_{bb}(\mathrm{SQU})]^{1/2}$ for $k = 8, 9, 10$ and hence

$$\sum_{k=7}^{10} G_{ab}(\beta(k)) \leqslant 4\varepsilon [G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}.$$

Hence by (4) and (5),

$$G_{ab}(Q) \leqslant \sum_{k=1}^{10} G_{ab}(\beta(k)) \leqslant 11\varepsilon [G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}$$

and Lemma 3 is proved.

To complete the proof of Lemma 0, observe that by Lemmas 1–3,

$$\begin{aligned}
&\int_0^1 \int_0^1 a(x)\, b(y)\, H(x, y)\, dy\, dx \\
&\qquad = \iint_R a(x)\, b(y)\, H(x, y)\, dy\, dx + \iint_Q a(x)\, b(y)\, H(x, y)\, dy\, dx \\
&\qquad \leqslant (t/u)\, G_{ab}(R) + G_{ab}(Q)
\end{aligned}$$

$$\leqslant (t/u)\, 2u^{-4}[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}$$
$$+ 11u^{1/6}[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}$$
$$= 13t^{1/31}[G_{aa}(\mathrm{SQU})\, G_{bb}(\mathrm{SQU})]^{1/2}.$$

Lemma 0 is proved. This completes the proof of Theorem 1.

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